Dual Dig Level I (2010) - Solutions

Unless otherwise stated or implied, assume that all graphs are in the usual *xy*-plane.

1. If p is a prime number greater than 2010, which of the following cannot be a prime number?

   A. $p+2$
   B. $p+100$
   C. $p+2009$
   D. $p+2010$

   Answer: C

   Explanation: A prime number p that is greater than 2010 must be odd. Therefore, $p+2009$ must be even, and no even number aside from 2 can be prime. Note: The answer can't be 'A' because, for example, 2027 and 2029 are both prime. The answer can't be 'B' because, for example, 2011 and 2111 are both prime. The answer can't be 'D' because, for example, 2011 and 4021 are both prime. (You weren't expected to notice any of those examples!)

2. The diameter of a circle is d meters. If the circle is enlarged so that the diameter is increased by π meters, what is the increase in the circumference of the circle?

   Answer: π^2 m, or π^2 meters

   Explanation: The circumference of the original circle is πd meters. The circumference of the enlarged circle is $\pi(d+\pi)$ meters, or $\pi d+\pi^2$ meters. Therefore, the increase in the circumference is π^2 meters.

3. What is the greatest odd factor of 6000?

   Answer: 375

   Explanation: The prime (or prime-power) factorization of 6000 is given by: $2^4 \cdot 3 \cdot 5^3$. Therefore, the greatest odd factor is given by: $3 \cdot 5^3 = 375$.

4. Simplify completely: $\dfrac{4x^{-2}-9y^{-2}}{10x^{-1}-15y^{-1}}$

Answer: $\dfrac{2y+3x}{5xy}$

Explanation: First, multiply by the LCD: $\dfrac{4x^{-2}-9y^{-2}}{10x^{-1}-15y^{-1}} \bullet \dfrac{x^2y^2}{x^2y^2}$. This results in: $\dfrac{4y^2-9x^2}{10xy^2-15x^2y}$.

Factoring yields: $\dfrac{(2y+3x)\cancel{(2y-3x)}}{5xy\cancel{(2y-3x)}}$, which simplifies to the answer, where $2y-3x \neq 0$.

5. Simplify completely: $\sqrt{x\left(\sqrt[3]{x\left(\sqrt[4]{x}\right)}\right)}$

Answer: $x^{\frac{17}{24}}$ or $\sqrt[24]{x^{17}}$

Explanation: Convert all radicals to rational exponents: $\left(x^1 \bullet \left(x^1 \bullet x^{\frac{1}{4}}\right)^{\frac{1}{3}}\right)^{\frac{1}{2}}$. Then, apply the rules of exponents: $\left(x^1 \bullet \left(x^{\frac{5}{4}}\right)^{\frac{1}{3}}\right)^{\frac{1}{2}} = \left(x^1 \bullet x^{\frac{5}{12}}\right)^{\frac{1}{2}} = \left(x^{\frac{17}{12}}\right)^{\frac{1}{2}} = x^{\frac{17}{24}}$.

6. The digits 1, 2, 3, 4, and 5 are placed in each row and column of a 5x5 matrix so that each column and each row contains only one of each of those digits. Which digit must be placed in the bottom right corner?

_	5	4	_	_
1	3	_	_	_
_	_	5	3	_
2	_	3	1	_
_	_	_	_	??

Answer: 3

Explanation:

3	5	4	2	1
1	3	2	5	4
4	1	5	3	2
2	4	3	1	5
5	2	1	4	[3]

7. Simplify completely: $3+\sqrt{3}+\frac{1}{3+\sqrt{3}}+\frac{1}{3-\sqrt{3}}$

Answer: $4+\sqrt{3}$

Explanation:

$$\begin{aligned}3+\sqrt{3}+\frac{1}{3+\sqrt{3}}+\frac{1}{3-\sqrt{3}} &= 3+\sqrt{3}+\frac{\left(3-\sqrt{3}\right)+\left(3+\sqrt{3}\right)}{\left(3+\sqrt{3}\right)\left(3-\sqrt{3}\right)}\\ &= 3+\sqrt{3}+\frac{6}{(3)^2-\left(\sqrt{3}\right)^2}\\ &= 3+\sqrt{3}+\frac{6}{9-3}\\ &= 3+\sqrt{3}+\frac{6}{6}\\ &= 3+\sqrt{3}+1\\ &= 4+\sqrt{3}\end{aligned}$$

8. What fraction of the first one million positive integers are perfect squares? Make sure your fraction is simplified.

Answer: $\frac{1}{1000}$

Explanation: Observe that $1{,}000{,}000=(1000)^2$. Therefore, exactly 1000 of the first one million positive integers are perfect squares: $(1)^2,(2)^2,\ldots,(1000)^2$. $\frac{1000}{1{,}000{,}000}=\frac{1}{1000}$.

9. Let $f(x)=\frac{x+1}{x-1}$. Evaluate and simplify $f(f(2010))$.

Answer: 2010

Explanation: $f(2010)=\frac{2010+1}{2010-1}=\frac{2011}{2009}$, and

$$f(f(2010))=f\left(\frac{2011}{2009}\right)=\frac{\frac{2011}{2009}+1}{\frac{2011}{2009}-1}=\frac{\frac{4020}{\cancel{2009}}}{\frac{2}{\cancel{2009}}}=\frac{4020}{2}=2010.$$

In fact, $f(f(x))=x$ for all real values of x except for 1. Work it out!

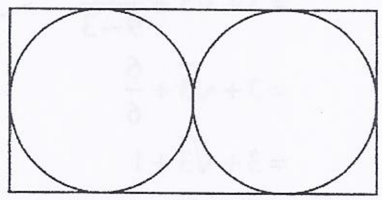
10. Evaluate $999999^2 - 999998^2$. (This can also be written as: $999{,}999^2 - 999{,}998^2$.)

Answer: 1,999,997

Explanation: We have a difference of two squares. Factor!

$$\begin{aligned}999{,}999^2 - 999{,}998^2 &= (999{,}999 + 999{,}998)(999{,}999 - 999{,}998)\\ &= (1{,}999{,}997)(1)\\ &= 1{,}999{,}997\end{aligned}$$

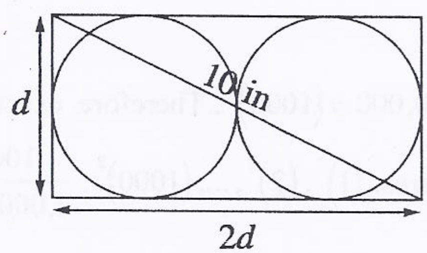
11. Two congruent circles lie tangent to each other. Together, they are circumscribed by a rectangle, as shown below (not to scale). If each diagonal of the rectangle is 10 inches, what is the area of the rectangle?



Answer: 40 in^2

Explanation: Let d be the diameter of each circle. The rectangle then has dimensions $2d \times d$. By the Pythagorean theorem, $(d)^2 + (2d)^2 = (10 \text{ in})^2 \Rightarrow 5d^2 = 100 \text{ in}^2 \Rightarrow d^2 = 20 \text{ in}^2$.

The area of the rectangle is given by: $(2d)(d) = 2d^2 = 2(20 \text{ in}^2) = 40 \text{ in}^2$.



12. If $x+1 = 10^{2010}$, then what real value of k satisfies $x^2 + 2x + 1 = 10^k$?

Answer: 4020

Explanation: If $x+1 = 10^{2010}$, then $x^2 + 2x + 1 = (x+1)^2 = (10^{2010})^2 = 10^{4020}$.

13. Larry and Margee can clean their entire house in 7 hours, while their toddler, Kristen, just by being around, can completely mess it up in only 2 hours. Larry and Margee completely clean their house while Kristen is at her grandparents, and they continue to clean the house once Kristen returns home. From the time Kristen returns home, how long will it be until the house is in complete shambles? (And don't ask why they don't put the kid in a playpen… obviously they're not that smart.)

Answer: 2.8 hours, or $2\frac{4}{5}$ hours, or $\frac{14}{5}$ hours

Explanation: Let x = the number of hours Margee, Larry, and Kristen were all doing their 'thing.' Note that Margee and Larry are doing 'negative work' relative to Kristen.

Using the model where: work = (rate)x(time elapsed), then: $\frac{-x}{7}+\frac{x}{2}=1$. Clearing fractions yields: $-2x+7x=14$, and the solution follows.

14. Find all real solutions of: $\log_{(x-4)}(17x-134)=2$

Answer: $x = 10$ or $x = 15$. That is, the solution set is: $\{10,15\}$.

Explanation: Use the definition of logarithms to rewrite the problem as:

$$\begin{gathered}(x-4)^2=17x-134\\ x^2-8x+16=17x-134\\ x^2-25x+150=0\\ (x-10)(x-15)=0\\ x=10 \quad \text{or} \quad x=15\end{gathered}$$

Note: Both solutions check out in the original equation.

15. Find an equation of the parabola in the usual *xy*-plane that passes through the points (0, – 4), (1, 5), and (– 3, – 7).

Answer: $y = 2x^2 + 7x - 4$ or $y = 2(x+\frac{7}{4})^2 - \frac{81}{8}$

Explanation: This sets up, initially, as a 3x3 system of equations, where each equation is of the form: $y = ax^2 + bx + c$. Then,

$(0,-4) \Rightarrow -4 = a(0)^2 + b(0) + c \Rightarrow c = -4$,

and $(1,5) \Rightarrow 5 = a(1)^2 + b(1) + c \Rightarrow a + b + c = 5$,

and $(-3,-7) \Rightarrow -7 = a(-3)^2 + b(-3) + c \Rightarrow 9a - 3b + c = -7$

Substitute $c = -4$ (from 1st equation) into the other two equations, simplify, and you get the new 2x2 system: $\begin{cases} a+b=9 \\ 9a-3b=-3 \end{cases}$, which can be solved (via substitution or elimination) to get $a = 2$ and $b = 7$.

Since we already know that $c = -4$, the desired equation is: $y = 2x^2 + 7x - 4$.

16. A chemist has two solutions of sulfuric acid. The first is half sulfuric acid and half water; the second is 75% sulfuric acid and 25% water. The chemist wishes to use a combination of the two solutions to make 10 liters of a new solution that is two-thirds sulfuric acid and one-third water. How many liters of the half-and-half solution should she use in her new solution?

Answer: $3\frac{1}{3}$ liters of the half-and-half mixture.

Explanation: Let x = number of liters of 75% mixture; let y = number of liters of 50% mixture.

Solve the system: $\begin{cases} x+y=10 \\ \frac{3}{4}x+\frac{1}{2}y=\frac{2}{3}(10) \end{cases}$ for y. Hint: Use the LCD, 12, to 'clear' fractions in the 2nd equation.

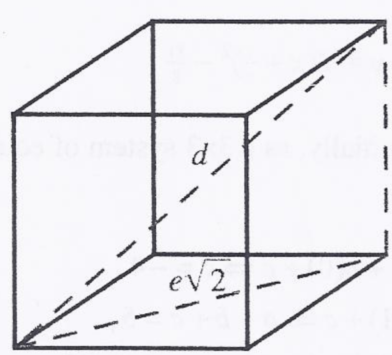
17. If the diagonal of a cube is d units long, find the <u>surface area</u> of the same cube, in terms of d.

Answer: $2d^2$ (square units)

Explanation: Let e equal the length of each edge of the cube. Then, the <u>area of one face</u> of the cube equals e^2, and the surface area of the cube equals $6e^2$. By the Pythagorean theorem, the length of a <u>diagonal of one face</u> of the cube equals $e\sqrt{2}$. By applying the Pythagorean theorem to the interior triangle (the dashed figure below): $d^2 = \left(e\sqrt{2}\right)^2 + e^2$. Simplifying, we get: $d^2 = 3e^2$. Then, $e^2 = \frac{d^2}{3}$.

Back-substitute into the expression for the surface area of the cube and we arrive at:

$SA = 6e^2 = 6\left(\frac{d^2}{3}\right) = 2d^2$ (square units).



18. Find all real solutions of: $4^{x+1}+16\left(4^{-x}\right)=65$

Answer: $x=-1$ or $x=2$. That is, the solution set is: $\{-1,2\}$.

Explanation: First use the rules of exponents to rewrite the problem as: $4^1 \bullet 4^x+\frac{16}{4^x}=65$. (It's easier at this step if you substitute, say, y for 4^x). Thus, $4y+\frac{16}{y}=65$. Multiply both sides by the LCD, y, to clear fractions:

$$4y^2+16=65y$$
$$4y^2-65y+16=0$$
$$(4y-1)(y-16)=0$$
$$y=\frac{1}{4} \quad \text{or} \quad y=16$$
$$4^x=\frac{1}{4} \quad \text{or} \quad 4^x=16$$
$$x=-1 \quad \text{or} \quad x=2$$

9. Find the exact area of the region bounded by the graphs of: $\begin{cases} x\sqrt{3}-y=4\sqrt{3} \\ x\sqrt{3}+y=12\sqrt{3} \\ y=0 \end{cases}$

Answer: $16\sqrt{3}$ units^2

Explanation:
1st – Do a quick sketch of each line: a triangle is formed.
2nd – Find the x-intercepts of the lines corresponding to the top two equations.
The x-intercepts are: (4, 0) and (12, 0), respectively.
We can treat the line segment connecting the two x-intercepts as the base of the triangle.
The base has length 8 units.
3rd – Find the point of intersection for the first two lines: $\left(8,4\sqrt{3}\right)$.

The height of the triangle is $4\sqrt{3}$ units.

4th – Find the area of the triangle: $A=\frac{1}{2}bh=\frac{1}{2}(8)\left(4\sqrt{3}\right)=16\sqrt{3}$ units^2.

20. Find the remainder when 3^{2010} is divided by 5.

Answer: 4

Explanation #1 (Identifying a pattern): Remember, when you divide by 5, the remainders must be either: 0, 1, 2, 3, or 4. (In fact, the remainder cannot be 0 for the powers of 3 corresponding to nonnegative integer exponents, because those powers of 3 will never be divisible by 5; the only prime number they are divisible by is 3.) Study the pattern that develops when successive powers of 3 are divided by 5:

(Power of 3) ÷ 5	remainder
$3^0 \div 5$	1
$3^1 \div 5$	3
$3^2 \div 5$	4
$3^3 \div 5$	2
$3^4 \div 5$	1
$3^5 \div 5$	3
$3^6 \div 5$	4
$3^7 \div 5$	2

Extend the powers of 3 and you will find that the remainders continue in exactly this pattern .... In number theory, this is called a modulus 4 cycle.

When 2010 is divided by 4, the remainder is 2. (In number theory, we write $2010 \equiv 2 (\text{mod}\, 4)$.)

This means that the remainder for 3^{2010} will be the same as the remainder for 3^2 when they are divided by 5.

Explanation #2 (Binomial Theorem):

$3^{2010} = 3^{2008} \cdot 3^2 = 3^{4 \cdot 502} \cdot 3^2 = \left(3^4\right)^{502} \cdot 3^2 = \left(81\right)^{502} \cdot 9 = \left(80+1\right)^{502} \cdot 9$.

If $\left(80+1\right)^{502}$ is expanded, perhaps by the Binomial Theorem, every term except for "+1" will be divisible by 80. Therefore, $\left(80+1\right)^{502} = 80k+1$ for some positive integer k. Now:

$\left(80+1\right)^{502} \cdot 9 = \left(80k+1\right) \cdot 9 = 720k+9 = 720k+5+4 = 5\left(144k+1\right)+4 = 5d+4$, where the positive integer $d = 144k+1$. Therefore, 3^{2010} yields a remainder of 4 when it is divided by 5.

MATH FIELD DAY 2010
SOLUTIONS FOR DUAL DIG I

E-copies of solutions (and an old Dual Dig II contest) will soon be placed on the following web site:

http://www.kkuniyuk.com

Feel free to download and use the various resources (including a Precalculus "book") available on the site. You may also send email through the site.

Thank you for participating!

San Diego Mesa College
(Ken Kuniyuki, Coordinator)